THE RULED V_4^4 IN S_5 ASSOCIATED WITH A SCHLÄFLI HEXAD*

BY

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1. **Introduction.** In S_4 we know that all the lines which meet four generic planes also meet a fifth associated plane. These ∞^2 lines generate a ruled V_3^3, i.e., the variety of Segre with 10 double points. This property cannot be generalized in space of more than four dimensions; that is: All the lines which meet n generic $(n-2)$-flats will not in general meet an additional $(n-2)$-flat when $n>4$; $n+1$ generic flats will determine $n+1$ V_{n-1}^{n-1}'s.

If however the $n+1$ S_{n-2}'s form a Schläfli set, a single V_{n-1}^{n-1} is determined. An equation of this spread has been given by C. R. Rupp.† Let the Schläfli set be given as follows‡:

$$x_i = 0, \quad \sum_0^n b_{ik}x_k = 0 \qquad (i = 0, 1, \cdots, n), \tag{1}$$

where $b_{ii}=0$, $b_{ik}=b_{ki}$. The equation of the V_{n-1}^{n-1} determined by the first n flats is then

$$\begin{vmatrix} b_{01} & b_{02} & b_{03} & \cdots & b_{0,n-1} & b_{0n} \\ -\sum_a^n b_{1i}x_i & b_{12}x_1 & b_{13}x_1 & \cdots & b_{1,n-1}x_1 & b_{1n}x_1 \\ b_{21}x_2 & -\sum_0^n b_{2i}x_i & b_{23}x_2 & \cdots & b_{2,n-1}x_2 & b_{2n}x_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ b_{n-1,1}x_n & b_{n-1,2}x_n & b_{n-1,3}x_n & \cdots & -\sum_0^n b_{n-1,i}x_i & b_{n-1,n}x_n \end{vmatrix} = 0. \tag{2}$$

It may easily be verified that the same V_{n-1}^{n-1} will be obtained by taking any other set of n flats from (1). All the $n+1$ flats lie on the spread and the fundamental (or regular) singular loci are $(n-2r)$-flats of multiplicity $r=2, 3, \cdots, n/2$, when n is even, and of multiplicity $r=2, 3, \cdots, (n-1)/2$, when n is odd; there are $\binom{n+1}{r}$ such loci. The study of the remaining accessory singular loci for the case $n=5$ will be the object of the present paper.

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† C. R. Rupp, *An extension of Pascal's theorem*, these Transactions, vol. 31, p. 578.

‡ Luigi Berzolari, *Sui sistemi di $n+1$ rette dello spazio ad n dimensioni, situate in posizione di Schläfli*, Rendiconti del Circolo Matematico di Palermo, vol. 20, pp. 229–247.

That this V_{n-1}^{n-1}* is not generic may be shown thus. By a projective transformation

$$x_0 = x_0', \quad x_i = x_i'/b_{0i} \qquad (i = 1, 2, \cdots, n),$$

a Schläfli set may be carried into the slightly modified form

$$x_0 = 0, \quad \sum_1^n x_k = 0; \quad x_i = 0, \quad x_0 + \sum b_{ik}x_k = 0 \qquad (i = 1, 2, \cdots, n),$$

where $b_{ii}=0$, $b_{i0}=1$, $b_{ik}=b_{ki}$, $k\neq i$. A general Schläfli set depends therefore on $n(n-1)/2$ parameters. Let there now be given $n+1$ generic S_{n-2}'s in S_n:

$$\sum_0^n a_i^{(1)} x_i = \sum_0^n b_i^{(1)} x_i = 0; \quad \sum_0^n a_i^{(2)} x_i = \sum_0^n b_i^{(2)} x_i = 0, \cdots,$$

$$\sum_0^n a_i^{(n+1)} x_i = \sum_0^n b_i^{(n+1)} x_i = 0,$$

which depend on $2(n-1)(n+1)$ parameters. A projective transformation can therefore be found which will reduce this number to

$$2(n - 1)(n + 1) - n(n + 2) = n^2 - 2n - 2.$$

But this number is greater than $n(n-1)/2$ when $n>4$, as we wished to prove.

2. **The equation of V_4^4 in Grassmann-Plücker coordinates.** The equation (2), being a determinant of the nth order, is rather unwieldy for the purpose of investigating the accessory singularities of a V_{n-1}^{n-1}; even in the case for $n=5$ the analytical work becomes formidable. We shall therefore use the Grassmann-Plücker coordinates and start with the equation of the generic V_4^4 which has been derived in a former paper (R, pp. 341–342): we shall then find the conditions which must be satisfied in order that it shall be associated with a Schläfli hexad†, and thus incidentally obtain the invariants of the spread. We shall suppose that V_4^4 has no triple point, that is, no three of the fundamental flats intersect. The equation of V_4^4 is‡

$$(3) \qquad V_4^4 = \begin{vmatrix} y_1 \sum_1^6 \alpha_{i1}y_i + y_2 \sum_1^6 \alpha_{i2}y_i & y_3 \sum_1^6 \alpha_{i3}y_i + y_5 \sum_1^6 \alpha_{i5}y_i \\ y_1 \sum_1^6 \beta_{i1}y_i + y_2 \sum_1^6 \beta_{i2}y_i & y_3 \sum_1^6 \beta_{i3}y_i + y_5 \sum_1^6 \beta_{i5}y_i \end{vmatrix} = 0,$$

* John Eiesland, *On a class of ruled* $(n-1)$*-spreads in* S_n, Rendiconti del Circolo Matematico di Palermo, vol. 54, pp. 335–365. By a "generic" V_{n-1}^{n-1} is meant here the V_{n-1}^{n-1} whose equation is given on p. 337. In what follows this paper will be referred to as "R." In this paper, the V_4^4, here denoted as the "generic V_4," is the generalization for $n=5$ of Segre's variety in S_4.

† By a Schläfli hexad in S_5 we mean here six 3-flats in Schläfli position.

‡ R, pp. 341–342.

or, if we add the elements of the first column to those of the second,

$$(3') \qquad V_4^4 = \begin{vmatrix} y_1 \sum_1^6 \alpha_{i1} y_i + y_2 \sum_1^6 \alpha_{i2} y_i & y_4 \sum_1^6 \alpha_{i4} y_i + y_6 \sum_1^6 \alpha_{i6} y_i \\ y_1 \sum_1^6 \beta_{i1} y_i + y_2 \sum_1^6 \beta_{i2} y_i & y_4 \sum_1^6 \beta_{i4} y_i + y_6 \sum_1^6 \beta_{i6} y_i \end{vmatrix} = 0.$$

The five fundamental flats are

$$y_1 = y_2 = 0; \quad y_3 = y_5 = 0; \quad y_4 = y_6 = 0; \quad \sum_1^6 a_i y_i = \sum_1^6 b_i y_i = 0;$$

$$\sum_1^6 c_i y_i = \sum_1^6 d_i y_i = 0, \quad \alpha_{ik} = a_i b_k - a_k b_i, \quad \beta_{ik} = c_i d_k - c_k d_i.$$

If now the V_4^4 belongs to a Schläfli hexad, all the lines which meet these five fundamental flats must also meet a sixth flat. In order to find such a flat we write (3) and (3′) as follows:

$$V_4^4 = \begin{vmatrix} y_1 M_1 + y_2 M_2 & y_3 M_3 + y_5 M_5 \\ y_1(L_1 + M_1) + y_2(L_2 + M_2) & y_3(L_3 + M_3) + y_5(L_5 + M_5) \end{vmatrix}$$

$$= \begin{vmatrix} y_1 M_1 + y_2 M_2 & y_4 M_4 + y_6 M_6 \\ y_1(L_1 + M_1) + y_2(L_2 + M_2) & y_4(L_4 + M_4) + y_6(L_6 + M_6) \end{vmatrix} = 0,$$

where $M_k = \sum \alpha_{ik} y_i$, $L_k = \sum \beta_{ik} y_i$, $k = 1, 2, \cdots, 6$. Consider the 3-flat

$$(4) \qquad y_1 + M_2 + L_2 = 0, \quad y_2 - M_1 - L_1 = 0.$$

If it is to be the required sixth flat it must be identical with the two flats

$$y_3 + M_3 + L_3 = 0, \ y_3 - M_3 - L_3 = 0; \ y_4 + M_6 + L_6 = 0, \ y_6 - M_4 - L_4 = 0.$$

If we set $P_{12} = 1 + \alpha_{12} + \beta_{12}$, $P_{35} = 1 + \alpha_{35} + \beta_{35}$, $P_{46} = 1 + \alpha_{46} + \beta_{46}$, and $(ik) = \alpha_{ik} + \beta_{ik}$, this means that the determinant

$$\begin{vmatrix} 0 & P_{12} & (13) & (15) & (14) & (16) \\ -P_{12} & 0 & (23) & (25) & (24) & (26) \\ (31) & (32) & 0 & P_{35} & (34) & (36) \\ (51) & (52) & -P_{35} & 0 & (54) & (56) \\ (41) & (42) & (43) & (45) & 0 & P_{46} \\ (61) & (62) & (63) & (65) & -P_{46} & 0 \end{vmatrix}$$

must be of rank 2. We thus obtain the following conditions:

$$
\begin{array}{ll}
 & P_{12}P_{35} = (13)(25) + (15)(32), \\
(5,\text{a}) & \begin{array}{ll} (36)P_{12} = (13)(26) + (16)(32), & (56)P_{12} = (15)(26) + (16)(52), \\ (34)P_{12} = (13)(24) + (14)(32), & (54)P_{12} = (15)(24) + (14)(52); \end{array}
\end{array}
$$

$$
\begin{array}{ll}
 & P_{35}P_{46} = (45)(36) + (34)(56), \\
(5,\text{b}) & \begin{array}{ll} (24)P_{35} = (23)(45) + (25)(34), & (14)P_{35} = (13)(45) + (15)(34), \\ (26)P_{35} = (23)(65) + (25)(36), & (16)P_{35} = (13)(65) + (15)(36); \end{array}
\end{array}
$$

$$
\begin{array}{ll}
 & P_{12}P_{46} = (16)(42) + (14)(26), \\
(5,\text{c}) & \begin{array}{ll} (23)P_{46} = (24)(36) + (26)(43), & (13)P_{46} = (14)(36) + (16)(43), \\ (25)P_{46} = (24)(56) + (26)(45), & (15)P_{46} = (14)(56) + (16)(45). \end{array}
\end{array}
$$

These relations are not independent; from any six of them the remaining ones may easily be derived. We also obtain the following important relations:

$$
\begin{array}{ll}
 & \dfrac{\alpha_{14}\beta_{16} - \alpha_{16}\beta_{14}}{\alpha_{15}\beta_{13} - \alpha_{13}\beta_{15}} = \dfrac{\alpha_{14}\beta_{26} - \alpha_{26}\beta_{14} + \alpha_{24}\beta_{16} - \alpha_{16}\beta_{24}}{\alpha_{25}\beta_{13} - \alpha_{13}\beta_{25} + \alpha_{15}\beta_{23} - \alpha_{23}\beta_{15}} = \dfrac{\alpha_{24}\beta_{26} - \alpha_{26}\beta_{24}}{\alpha_{25}\beta_{23} - \alpha_{23}\beta_{25}} = 1, \\[2ex]
(6) & \dfrac{\alpha_{14}\beta_{24} - \alpha_{24}\beta_{14}}{\alpha_{54}\beta_{34} - \alpha_{34}\beta_{54}} = \dfrac{\alpha_{14}\beta_{26} - \alpha_{26}\beta_{14} + \alpha_{16}\beta_{24} - \alpha_{24}\beta_{16}}{\alpha_{36}\beta_{45} - \alpha_{45}\beta_{36} + \alpha_{56}\beta_{34} - \alpha_{34}\beta_{56}} = \dfrac{\alpha_{16}\beta_{26} - \alpha_{26}\beta_{16}}{\alpha_{56}\beta_{36} - \alpha_{36}\beta_{56}} = 1, \\[2ex]
 & \dfrac{\alpha_{34}\beta_{36} - \alpha_{36}\beta_{34}}{\alpha_{23}\beta_{13} - \alpha_{13}\beta_{23}} = \dfrac{\alpha_{34}\beta_{56} - \alpha_{56}\beta_{34} + \alpha_{36}\beta_{45} - \alpha_{45}\beta_{36}}{\alpha_{25}\beta_{13} - \alpha_{13}\beta_{25} + \alpha_{23}\beta_{15} - \alpha_{15}\beta_{23}} = \dfrac{\alpha_{45}\beta_{65} - \alpha_{65}\beta_{45}}{\alpha_{25}\beta_{15} - \alpha_{15}\beta_{25}} = 1;
\end{array}
$$

$$
(7) \quad 1 + \Sigma\alpha = 1 + \alpha_{12} + \alpha_{35} + \alpha_{46} = 0, \quad 1 + \Sigma\beta = 1 + \beta_{12} + \beta_{35} + \beta_{46} = 0.
$$

The last two relations are obtained by using (5,b) and (6). Since there are six independent non-homogeneous relations between the 16 parameters of a generic V_4^4, the V_4^4 belonging to a Schläfli hexad has 10 essential parameters, as was shown before (p. 316) by a different method.

3. **The singular loci on the V_4^4 associated with a Schläfli hexad.** We know that the generic V_4^4 in S_5 has 5 fundamental 3-flats and that the singular loci lie in each of these flats.* In any one flat we have four fundamental double lines which are the intersections of the flat with the remaining four flats; moreover, two accessory double lines which intersect these in 8 points and, finally, a cubic curve which has the four fundamental lines as bisecants. Through each of the 40 points pass two double lines of which one is accessory, and one cubic; but it is to be noted that this cubic does not belong to the fundamental flat in which the accessory line is immersed. We have thus 20 lines and 5 cubics as the complete set of double loci on a generic V_4^4.

* R, pp. 344–352.

In the case at hand the V_4^4 has six fundamental flats in a Schläfli position. In each flat are five fundamental double lines and two accessory double lines which intersect these in 10 points. The cubic is composite, consisting of three lines, namely the fifth double line which is added to the four of the generic case, and two lines which remain. We shall prove that these two lines coincide with the two accessory lines. It will be sufficient to prove this for any one flat, since what happens in one must happen in all the other five flats.

The singular loci in the 3-flat $y_3=y_5=0$ are the complete intersection of the two cubic surfaces

$$y_3 = y_5 = 0, \quad \frac{\partial V_4^4}{\partial y_3} = 0, \quad \frac{\partial V_4^4}{\partial y_5} = 0.$$

We have then, from (3),

$$\tag{8} \begin{aligned} \phi_1 &= \frac{\partial V_4^4}{\partial y_3} = P \sum_{1,2}^{4,6} \beta_{i3} y_i - Q \sum_{1,2}^{4,6} \alpha_{i3} y_i = 0, \\ \phi_2 &= \frac{\partial V_4^4}{\partial y_5} = P \sum_{1,2}^{4,6} \beta_{i5} y_i - Q \sum_{1,2}^{4,6} \alpha_{i5} y_i = 0. \end{aligned}$$

The four fundamental double lines are

$$\tag{9} \begin{aligned} &y_3 = y_5 = y_1 = y_2 = 0, \quad y_3 = y_5 = y_4 = y_6 = 0, \\ &y_3 = y_6 = M_3 = M_5 = 0, \; y_3 = y_5 = L_3 = L_5 = 0, \end{aligned}$$

to which must be added the fifth double line

$$\tag{10} y_3 = y_5 = L_3 + M_3 = L_5 + M_5 = 0,$$

which is the intersection of the flat $y_3=y_5=0$ with the fifth fundamental flat $y_3+M_3+L_3=0$, $y_5-M_3-L_3=0$. The two accessory double lines are

$$\tag{11} y_1 = \kappa y_2, \quad y_4 = \mu y_6,$$

where κ and μ are roots of the two quadratic equations

$$\tag{12} \begin{aligned} &(\alpha_{14}\beta_{16} - \alpha_{16}\beta_{14})\kappa^2 + (\alpha_{16}\beta_{42} - \alpha_{42}\beta_{16} + \alpha_{26}\beta_{41} - \alpha_{41}\beta_{26})\kappa \\ &\qquad\qquad + (\alpha_{26}\beta_{42} - \alpha_{42}\beta_{26}) = 0, \\ &(\alpha_{14}\beta_{24} - \alpha_{24}\beta_{14})\mu^2 + (\alpha_{24}\beta_{61} - \alpha_{61}\beta_{24} + \alpha_{26}\beta_{41} - \alpha_{41}\beta_{26})\mu \\ &\qquad\qquad + (\alpha_{24}\beta_{61} - \alpha_{61}\beta_{24}) = 0. \end{aligned}$$

These equations may also be written

$$\tag{13} \kappa = \frac{\alpha_{24}\mu + \alpha_{26}}{\alpha_{41}\mu + \alpha_{61}} = \frac{\beta_{24}\mu + \beta_{26}}{\beta_{41}\mu + \beta_{61}}, \qquad \mu = \frac{\alpha_{16}\kappa + \alpha_{26}}{\alpha_{41}\kappa + \alpha_{42}} = \frac{\beta_{16}\kappa + \beta_{26}}{\beta_{41}\kappa + \beta_{42}}.$$

The cubic curve being composite, consisting of the fifth double line and two additional lines, it is to be proved that these latter coincide with the two accessory lines (11); in other words, these lines are tac-loci on the two surfaces $\phi_1=\phi_2=0$. Take any point on the line (11), say $(\kappa\rho, \rho, \mu, 1)$. The tangent planes to the cubics at the point are

$$\left(\frac{\partial\phi_i}{\partial y_1}\right)y_1+\left(\frac{\partial\phi_i}{\partial y_2}\right)y_2+\left(\frac{\partial\phi_i}{\partial y_4}\right)y_4+\left(\frac{\partial\phi_i}{\partial y_6}\right)y_6=0 \quad (i=1,2).$$

Noting that the accessory lines are generators of the two quadrics

$$P = y_1(\alpha_{41}y_4+\alpha_{61}y_6)+y_2(\alpha_{42}y_4+\alpha_{62}y_6)=0,$$
$$Q = y_1(\beta_{41}y_4+\beta_{61}y_6)+y_2(\beta_{42}y_4+\beta_{62}y_6)=0,$$

we have

$$\left(\frac{\partial\phi_1}{\partial y_1}\right)=(\alpha_{41}\mu+\alpha_{61})L_3-(\beta_{41}\mu+\beta_{61})M_3,\quad \left(\frac{\partial\phi_1}{\partial y_2}\right)=(\alpha_{42}\mu+\alpha_{62})L_3-(\beta_{42}\mu+\beta_{62})M_3,$$
$$\frac{1}{\rho}\left(\frac{\partial\phi_1}{\partial y_4}\right)=(\alpha_{41}\kappa+\alpha_{42})L_3-(\beta_{41}\kappa+\beta_{42})M_3,\quad \frac{1}{\rho}\left(\frac{\partial\phi_1}{\partial y_6}\right)=(\alpha_{61}\kappa+\alpha_{62})L_3-(\beta_{61}\kappa+\beta_{62})M_3,$$
$$\left(\frac{\partial\phi_2}{\partial y_1}\right)=(\alpha_{41}\mu+\alpha_{61})L_5-(\beta_{41}\mu+\beta_{61})M_5,\quad \left(\frac{\partial\phi_2}{\partial y_2}\right)=(\alpha_{42}\mu+\alpha_{62})L_5-(\beta_{42}\mu+\beta_{62})M_5,$$
$$\frac{1}{\rho}\left(\frac{\partial\phi_2}{\partial y_4}\right)=(\alpha_{41}\kappa+\alpha_{42})L_5-(\beta_{41}\kappa+\beta_{42})M_5,\quad \frac{1}{\rho}\left(\frac{\partial\phi_2}{\partial y_6}\right)=(\alpha_{61}\kappa+\alpha_{62})L_5-(\beta_{61}\kappa+\beta_{62})M_5,$$

where we have set

$$L_3=(\beta_{13}\kappa+\beta_{23})\rho+\beta_{43}\mu+\beta_{63},\quad M_3=(\alpha_{13}\kappa+\alpha_{23})\rho+\alpha_{43}\mu+\alpha_{63},$$
$$L_5=(\beta_{15}\kappa+\beta_{25})\rho+\beta_{45}\mu+\beta_{65},\quad M_5=(\alpha_{15}\kappa+\alpha_{25})\rho+\alpha_{45}\mu+\alpha_{65}.$$

If now the line (11) is to be a tac-locus on $\phi_1=\phi_2=0$ we must have

$$\frac{\left(\frac{\partial\phi_1}{\partial y_1}\right)}{\left(\frac{\partial\phi_2}{\partial y_1}\right)}=\frac{\left(\frac{\partial\phi_1}{\partial y_2}\right)}{\left(\frac{\partial\phi_2}{\partial y_2}\right)}=\frac{\left(\frac{\partial\phi_1}{\partial y_4}\right)}{\left(\frac{\partial\phi_2}{\partial y_4}\right)}=\frac{\left(\frac{\partial\phi_1}{\partial y_6}\right)}{\left(\frac{\partial\phi_2}{\partial y_6}\right)}$$

$$=\frac{L_3-p_1M_3}{L_5-p_1M_5}=\frac{L_3-p_2M_3}{L_5-p_2M_5}=\frac{L_3-p_3M_3}{L_5-p_3M_5}=\frac{L_3-p_4M_3}{L_5-p_4M_5},$$

where

$$p_1 = \frac{\beta_{41}\mu + \beta_{61}}{\alpha_{41}\mu + \alpha_{61}}, \quad p_2 = \frac{\beta_{42}\mu + \beta_{62}}{\alpha_{42}\mu + \alpha_{62}}, \quad p_3 = \frac{\beta_{41}\kappa + \beta_{42}}{\alpha_{41}\kappa + \alpha_{42}}, \quad p_4 = \frac{\beta_{61}\kappa + \beta_{62}}{\alpha_{61}\kappa + \alpha_{62}}.$$

But the equations (13) show that $p_1 = p_2$ and $p_3 = p_4$, hence we get the single condition

$$\frac{L_3 - p_1 M_3}{L_5 - p_1 M_5} = \frac{L_3 - p_3 M_3}{L_5 - p_3 M_5}, \quad p_1 - p_3 \neq 0,$$

which is equivalent to the condition $L_3 M_5 - L_5 M_3 = 0$, true for all values of ρ. We thus obtain the following three relations:

$$(14)\quad \begin{aligned} (\beta_{13}\kappa + \beta_{23})(\alpha_{15}\kappa + \alpha_{25}) - (\beta_{15}\kappa + \beta_{25})(\alpha_{13}\kappa + \alpha_{23}) &= 0, \\ (\beta_{43}\mu + \beta_{63})(\alpha_{45}\mu + \alpha_{65}) - (\beta_{45}\mu + \beta_{65})(\alpha_{43}\mu + \alpha_{63}) &= 0, \\ (\beta_{13}\kappa + \beta_{23})(\alpha_{45}\mu + \alpha_{65}) - (\alpha_{15}\kappa + \alpha_{25})(\beta_{43}\mu + \beta_{63}) & \\ - (\beta_{15}\kappa + \beta_{25})(\alpha_{43}\mu + \alpha_{63}) - (\alpha_{13}\kappa + \alpha_{23})(\beta_{45}\mu + \beta_{65}) &= 0. \end{aligned}$$

The third relation is satisfied by virtue of the first two, hence κ and μ must be roots of the two quadratic equations

$$(15)\quad \begin{aligned} (\alpha_{15}\beta_{13} - \beta_{15}\alpha_{13})\kappa^2 + (\alpha_{15}\beta_{23} - \alpha_{23}\beta_{15} + \alpha_{25}\beta_{13} - \alpha_{13}\beta_{25})\kappa & \\ + \alpha_{25}\beta_{23} - \beta_{25}\alpha_{23} &= 0, \\ (\alpha_{45}\beta_{43} - \alpha_{43}\beta_{45})\mu^2 + (\alpha_{45}\beta_{63} - \alpha_{63}\beta_{45} + \alpha_{65}\beta_{43} - \alpha_{43}\beta_{65})\mu & \\ + \alpha_{65}\beta_{63} - \alpha_{63}\beta_{65} &= 0. \end{aligned}$$

But κ and μ are roots of the quadratic equation (12), hence we must have

$$\frac{\alpha_{14}\beta_{16} - \alpha_{16}\beta_{14}}{\alpha_{15}\beta_{13} - \alpha_{13}\beta_{15}} = \frac{\alpha_{16}\beta_{42} - \alpha_{42}\beta_{16} + \alpha_{26}\beta_{41} - \alpha_{41}\beta_{26}}{\alpha_{15}\beta_{23} - \alpha_{23}\beta_{15} + \alpha_{25}\beta_{13} - \alpha_{13}\beta_{25}} = \frac{\alpha_{26}\beta_{42} - \alpha_{42}\beta_{26}}{\alpha_{25}\beta_{23} - \alpha_{23}\beta_{25}},$$

$$\frac{\alpha_{14}\beta_{24} - \alpha_{24}\beta_{14}}{\alpha_{45}\beta_{43} - \alpha_{43}\beta_{45}} = \frac{\alpha_{24}\beta_{61} - \alpha_{61}\beta_{24} + \alpha_{26}\beta_{41} - \alpha_{41}\beta_{26}}{\alpha_{45}\beta_{63} - \alpha_{63}\beta_{45} + \alpha_{65}\beta_{43} - \beta_{65}\alpha_{43}} = \frac{\alpha_{26}\beta_{61} - \alpha_{61}\beta_{26}}{\alpha_{65}\beta_{63} - \alpha_{63}\beta_{65}},$$

which are true according to equations (6), as we wished to prove.

If we set $\lambda = (\alpha_{15}\kappa + \alpha_{25})/(\alpha_{31}\kappa + \alpha)_{32}$, we have from (13) and (14),

$$(16)\quad \lambda = \frac{\alpha_{15}\kappa + \alpha_{25}}{\alpha_{31}\kappa + \alpha_{32}} = \frac{\beta_{15}\kappa + \beta_{25}}{\beta_{31}\kappa + \beta_{32}} = \frac{\alpha_{45}\mu + \alpha_{65}}{\alpha_{34}\mu + \alpha_{36}} = \frac{\beta_{45}\mu + \beta_{65}}{\beta_{34}\mu + \beta_{36}},$$

so that λ must be a root of the two equations

$$(17)\quad \begin{aligned} (\alpha_{23}\beta_{31} - \alpha_{31}\beta_{23})\lambda^2 + (\alpha_{23}\beta_{51} - \alpha_{51}\beta_{23} + \alpha_{25}\beta_{31} - \alpha_{31}\beta_{25})\lambda & \\ + \alpha_{25}\beta_{51} - \alpha_{51}\beta_{25} &= 0, \\ (\alpha_{63}\beta_{34} - \alpha_{34}\beta_{63})\lambda^2 + (\alpha_{63}\beta_{54} - \alpha_{54}\beta_{63} + \alpha_{65}\beta_{34} - \alpha_{34}\beta_{65})\lambda & \\ + \alpha_{65}\beta_{54} - \alpha_{54}\beta_{65} &= 0, \end{aligned}$$

which have identical roots, equation (6).

We have thus $6\cdot 5/2=15$ fundamental double lines and 12 accessory double lines; these lines intersect in 30 points, three lines through each point. No point is a triple point. We shall prove the following important

THEOREM. *A V_4^4 in S_5, associated with a Schläfli hexad, has two double planes. The* 12 *accessory lines form a double-six, each set of six lines forming a complete hexagon in each plane. The* 15 *fundamental double lines are outside of these planes and join the* 15 *pairs of corresponding vertices of the hexagon. The equations of the planes are*

$$y_1 = \kappa y_2, \qquad y_3 = \lambda y_5, \qquad y_4 = \mu y_6, \tag{18}$$

where κ, λ and μ are the roots of the three quadratic equations (12) *and* (17).

The proof of the first part of this theorem is immediate. If we substitute the values of y_1, y_2, and y_3 from (18) in the equation (3), the determinant reduces to one of rank zero, since all the elements vanish, account being taken of (13) and (16); to prove the second part we need only show that any one of the planes contains the six accessory lines

$$\begin{aligned}
&y_1 = 0, \quad y_2 = 0, \qquad y_3 = \lambda y_5, \qquad y_4 = \mu y_6,\\
&y_3 = 0, \quad y_5 = 0, \qquad y_1 = \kappa y_2, \qquad y_4 = \mu y_6,\\
&y_4 = 0, \quad y_6 = 0, \qquad y_1 = \kappa y_2, \qquad y_3 = \lambda y_5;
\end{aligned} \tag{19,a}$$

$$\begin{aligned}
&y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad \sum \alpha_{i3} y_i = 0, \quad \sum \alpha_{i5} y_i = 0,\\
&y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad \sum \beta_{i3} y_i = 0, \quad \sum \beta_{i5} y_i = 0,\\
&y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad y_1 + M_2 + L_2 = 0, \quad y_2 - M_1 - L_1 = 0.
\end{aligned} \tag{19,b}$$

The second set (19, b) may also be written

$$y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad y_4 = \mu y_6, \quad y_2 = \frac{\alpha_{35} y_5}{\alpha_{13}\kappa + \alpha_{23}} + \frac{\alpha_{46} y_6}{\alpha_{14}\kappa + \alpha_{24}},$$

$$y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad y_4 = \mu y_6, \quad y_2 = \frac{\beta_{35} y_5}{\beta_{13}\kappa + \beta_{23}} + \frac{\beta_{46} y_6}{\beta_{14}\kappa + \beta_{24}},$$

$$y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad y_4 = \mu y_6, \quad y_2 = \frac{P_{35} y_3}{(13)\kappa + (23)} + \frac{P_{46} y_6}{(14)\kappa + (24)}.$$

From these equations it is at once evident that on each of the planes (18) the six accessory lines form a complete hexagon, and that any one of the 15 fundamental double lines joins a pair of corresponding vertices of the two hexagons.

THEOREM. *The* 12 *sides of the two hexagons are bispatial.*

To prove this we transform the origin (0, 0, 0, 0, 0, 1) to the point ($\rho\kappa$, ρ, $\sigma\lambda$, μ, σ, 1) on one of the double planes. The tangent cone, the equation of which being rather long, we shall not give here, is seen to be reducible for the following values of ρ and σ:

$$\rho = 0, \quad \sigma = 0, \quad \rho_1 = \frac{\alpha_{35}\sigma}{\alpha_{13}\kappa + \alpha_{23}} + \frac{\alpha_{46}}{\alpha_{14}\kappa + \alpha_{24}}, \quad \rho_2 = \frac{\beta_{35}\sigma}{\beta_{13}\kappa + \beta_{23}} + \frac{\alpha_{46}}{\beta_{14}\kappa + \beta_{24}},$$

$$\rho_3 = \frac{P_{35}\sigma}{(13)\kappa + (23)} + \frac{P_{46}}{(14)\kappa + (24)}.$$

These five values correspond to five sides of the hexagon. That the sixth side, $y_1 = \kappa y_2$, $y_3 = \lambda y_5$, $y_4 = y_6 = 0$, is also bispatial is proved by transforming the origin to the point ($\rho\kappa$, ρ, λ, $\mu\tau$, 1, τ). The cone is reducible when $\tau = 0$.

From these two theorems we derive the following

COROLLARY. *Given in S_5 a hexad of 3-flats in a Schläfli position. There exist two planes which intersect these flats in* 12 *lines forming a complete hexagon in each plane. The V_4^4 associated with the hexad has the two planes for double planes and the* 15 *lines joining the corresponding vertices of the hexagons are double lines on the V_4^4. The* 12 *lines of intersection are bispatial.*

4. **Transformation of the V_4^4.** In order to carry the V_4^4 into the form given by equation (2) for $n = 5$ we set up the following transformation:

$$x_0 = y_1, \quad \sum_0^5 b_{0i}x_i = y_2, \quad \sum_0^5 b_{1i}x_i = y_5, \quad x_2 = y_4, \quad \sum_1^5 b_{2i}x_i = y_6, \quad x_1 = y_3, \tag{20}$$

from which it must follow that

$$\sum_1^6 a_i y_i = x_3, \quad \sum b_i y_i = \sum_0^5 b_{3i}x_i, \quad \sum_1^6 c_i y_i = x_4, \quad \sum_1^6 d_i y_i = \sum b_{4i}x_i. \tag{21}$$

Substituting the values of y_i from (20) on the left side of these equations and comparing the coefficients of the x's on both sides we find the values of a_i, b_i, c_i, d_i expressed rationally in terms of the b_{ik}. We then calculate the Grassmann coordinates α_{ik}, β_{ik}. The work is rather long and tedious, but affords a valuable check on the correctness of the method we have pursued; in fact, *the α_{ik}, β_{ik} thus found are seen to satisfy the fundamental relations* (5,a), (5,b), *and* (5,c).

The singular loci of the V_4^4 having been found, our work is completed. If we had started with the equation (2) for $n = 5$ we should have failed, the analytic work being too complicated.

5. The self-dual V_4^4 associated with a Schläfli hexad. In a former paper* it was proved that if on a generic V_4^4 in S_5 any two of the 10 fundamental double lines are bispatial, they are all bispatial and the spread is self-dual. In the case of the V_4^4 here considered a similar theorem holds: *If any one of the* 15 *fundamental double lines is bispatial, they are all bispatial and the* V_4^4 *is self-dual.*

Let the double line be $y_1=y_2=y_3=y_5=0$; transforming the origin to a point ρ on this line we set $y_i=y_i'$, $y_4=y_4'+\rho y_6'$, $i=1, 2, 3, 5, 6$. The equation (3) may then be written, dropping the primes,

$$\phi_2 y_6^2 + \phi_3 y_6 + \phi_4 = 0,$$

where

$$\begin{aligned}\phi_2 = &\ [(\alpha_{61} + \rho\alpha_{41})y_1 + (\alpha_{62} + \rho\alpha_{42})y_2][(\beta_{63} + \rho\beta_{43})y_3 + (\beta_{65} + \rho\beta_{45})y_5] \\ &- [(\beta_{61} + \rho\beta_{41})y_4 + (\beta_{32} + \rho\beta_{42})y_2][(\alpha_{63} + \rho\alpha_{43})y_3 + (\alpha_{65} + \rho\alpha_{45})y_5].\end{aligned}$$

In order that the point ρ shall be bispatial the discriminant of this form must be of rank 2. We have

$$\begin{aligned}\Delta = &\ [(\alpha_{65} + \rho\alpha_{45})(\beta_{63} + \rho\beta_{43}) - (\alpha_{63} + \rho\alpha_{43})(\beta_{65} + \rho\beta_{45})][(\beta_{61} + \rho\beta_{41})(\alpha_{62} + \rho\alpha_{42}) \\ &- (\beta_{62} + \rho\beta_{42})(\alpha_{61} + \rho\alpha_{41})] = 0.\end{aligned}$$

Since every point ρ is to be bispatial we must have

$$\alpha_{45}\beta_{43} - \alpha_{43}\beta_{45} = \alpha_{42}\beta_{41} - \alpha_{41}\beta_{42} = 0, \quad \alpha_{65}\beta_{63} - \alpha_{63}\beta_{65} = \alpha_{62}\beta_{61} - \alpha_{61}\beta_{62} = 0,$$

$$\alpha_{65}\beta_{43} - \alpha_{43}\beta_{65} + \alpha_{45}\beta_{63} - \alpha_{63}\beta_{45} = 0, \quad \alpha_{42}\beta_{61} - \alpha_{61}\beta_{42} + \alpha_{62}\beta_{41} - \alpha_{41}\beta_{62} = 0,$$

which may also be written

$$(22,\text{a})\quad \frac{\beta_{41}}{\alpha_{41}} = \frac{\beta_{42}}{\alpha_{42}} = \frac{\beta_{61}}{\alpha_{61}} = \frac{\beta_{62}}{\alpha_{62}}, \qquad (22,\text{b})\quad \frac{\beta_{45}}{\alpha_{45}} = \frac{\beta_{43}}{\alpha_{43}} = \frac{\beta_{65}}{\alpha_{65}} = \frac{\beta_{63}}{\alpha_{63}};$$

hence three conditions must be satisfied, the second set being identical with the first, as follows from (6). If also the double line $y_3=y_5=0$, $y_4=y_6=0$ is bispatial, we get the following two sets of conditions:

$$(22,\text{c})\quad \frac{\beta_{13}}{\alpha_{13}} = \frac{\beta_{15}}{\alpha_{15}} = \frac{\beta_{23}}{\alpha_{23}} = \frac{\beta_{25}}{\alpha_{25}}, \qquad (22,\text{d})\quad \frac{\beta_{41}}{\alpha_{41}} = \frac{\beta_{61}}{\alpha_{61}} = \frac{\beta_{42}}{\alpha_{42}} = \frac{\beta_{62}}{\alpha_{62}};$$

that is, no new conditions are added, if account is taken of equations (6). It will not be necessary to carry out the work for the 13 remaining double lines as no new conditions are found. If now we set

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* R, pp. 358–360.

$$\frac{\beta_{41}}{\alpha_{41}} = \frac{\beta_{42}}{\alpha_{42}} = \frac{\beta_{61}}{\alpha_{61}} = \frac{\beta_{62}}{\alpha_{62}} = r, \qquad \frac{\beta_{13}}{\alpha_{13}} = \frac{\beta_{15}}{\alpha_{15}} = \frac{\beta_{23}}{\alpha_{23}} = \frac{\beta_{25}}{\alpha_{25}} = s,$$

$$\frac{\beta_{45}}{\alpha_{45}} = \frac{\beta_{43}}{\alpha_{43}} = \frac{\beta_{65}}{\alpha_{65}} = \frac{\beta_{63}}{\alpha_{63}} = t,$$

the equation of the V_4^4 in y-coordinates is

$$\begin{vmatrix} Q_2 + Q_3 & Q_1 - Q_3 \\ rQ_2 + sQ_3 & tQ_1 - sQ_3 \end{vmatrix} = 0, \tag{23}$$

or, when expanded,

$$\begin{aligned} &(r - s)Q_2Q_3 + (t - s)Q_1Q_3 + (t - r)Q_1Q_2 = 0, \\ &Q_1 = y_3(\alpha_{43}y_4 + \alpha_{63}y_6) + y_5(\alpha_{45}y_4 + \alpha_{65}y_6), \\ &Q_2 = y_1(\alpha_{41}y_4 + \alpha_{61}y_6) + y_2(\alpha_{42}y_4 + \alpha_{62}y_6), \\ &Q_3 = y_1(\alpha_{31}y_3 + \alpha_{51}y_5) + y_2(\alpha_{32}y_3 + \alpha_{52}y_5). \end{aligned} \tag{23'}$$

The equation of the V_4^4 in tangential coordinates is

$$\begin{aligned} &[(t - r)U_3 + (s - t)U_2 + (s - r)U_1]^2 - 4(s - t)(s - r)U_1U_2 = 0, \\ &U_1 = \frac{u_3(\alpha_{65}u_4 + \alpha_{54}u_6) + u_5(\alpha_{43}u_6 + \alpha_{36}u_4)}{\alpha_{46}\alpha_{35}}, \\ &U_2 = \frac{u_1(\alpha_{62}u_4 + \alpha_{24}u_6) + u_2(\alpha_{41}u_6 + \alpha_{16}u_4)}{\alpha_{46}\alpha_{12}}, \\ &U_3 = \frac{u_1(\alpha_{52}u_3 + \alpha_{23}u_5) + u_2(\alpha_{31}u_5 + \alpha_{15}u_3)}{\alpha_{35}\alpha_{12}}. \end{aligned} \tag{24}$$

Hence the order and class of V_4^4 are equal, as we wished to prove.

6. **The singularities of the self-dual V_4^4.** Since the conditions (22,a) and the resulting conditions (22,b), (22,c), and (22,d) imply that the three equations (15) and (17) are indeterminate, it follows that there will be an infinite number of double planes instead of only two. These planes are the generators of the 3-dimensional quadric $Q_1 = Q_2 = Q_3 = 0$, which may also be written

$$\frac{y_1}{y_2} = \frac{\alpha_{42}y_4 + \alpha_{62}y_6}{\alpha_{14}y_4 + \alpha_{16}y_6} = \frac{\alpha_{32}y_3 + \alpha_{52}y_5}{\alpha_{13}y_3 + \alpha_{15}y_5}.$$

Setting each of these ratios equal to a variable α we have the ∞^1 generating planes. The six fundamental flats intersect this quadric in six 2-dimensional quadrics which are all bispatial. We may therefore state the following theorem:

*The self-dual V_4^4 associated with a Schläfli hexad has a 3-dimensional quadric as locus of double points. The six fundamental flats intersect this quadric in six 2-dimensional quadrics which are all bispatial. There are ∞^7 such self-dual V_4^4's.**

* In R, p. 359, the fact was overlooked that the 2-dimensional quadric $x_1/x_2=x_3/x_5=x_4/x_6$ is a double locus on the self-dual V_4^4. We have then five 2-dimensional bispatial quadrics, one in each of the five fundamental flats.

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